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Abstract: Based on kinetic formulation for scalar conservation laws we present implicit kinetic schemes. For timestepping the schemes require resolution of linear systems of algebraic equations. We justify that the developed implicit framework is very suitable for steady state calculations. Namely, we prove the convergence towards steady state when t tends to infinity. To our knowledge this is the first theoretical result of this type for nonlinear scalar conservation laws. Then for the equation with stiff source term we construct a stiff numerical scheme with discontinuous coefficients that ensure the scheme to be equilibrium conserving. We couple the developed implicit approach with the stiff space discretization thus providing improved stability and equilibrium conservation property in the resulting scheme.

Numerical results demonstrate high computational capabilities (stability for large CFL numbers, fast convergence, accuracy) of the developed implicit approach.

Key-words: scalar conservation laws, kinetic formulation, stiff source terms, steady states, convergence

(Résumé : tsvp)

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Schémas Cinétiques Implicites pour les Lois de Conservation Scalaires

Résumé : Nous avons développé des schémas cinétiques implicites fondés sur la formulation cinétique des lois de conservation scalaires. Nous montrons la convergence vers une solution entropique stationnaire dans le cas des lois de conservation homogènes. Dans le cas des lois de conservation avec terme source, nous couplons l'approximation implicite avec une approximation raide des dérivées partielles en espace. Des résultats numériques montrent que le schéma cinétique implicite est stable et précis pour des grands CFL.

Mots-clé : lois de conservation hyperboliques, formulation cinétique, termes sources raides, solutions stationaires, convergence

Contents

1 Introduction

For hyperbolic conservation laws using of explicit finite difference schemes is very convenient because of their simplicity and accuracy at the same time; but their computational efficiency is substantially reduced by crucial CFL condition that restricts very much the size of time discretization step with the purpose of ensuring the stability for the scheme. Notice that small time step is very undesirable property, especially, for steady state calculations while using time marching procedure. On the opposite, purely implicit finite difference schemes enable application of sufficiently large CFL numbers maintaining at the same time stability of computations. But finding of solutions of nonlinear finite difference equations are very expensive from viewpoint of computational costs that restricts the efficiency of the implicit approach. In this paper we develop a special class of implicit kinetic schemes for scalar conservation laws

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial A_i(u)}{\partial x_i} + \nabla z(x) b(u) = 0, \quad t \geq 0, \ x \in \mathrm{I\!R}^{\mathrm{N}}, \tag{1.1}$$

$$u(x, t=0) = u_0(x), \quad u_0(x) \in L^1 \cap L^\infty \cap BV(\mathrm{I\!R}^{\mathrm{N}}), \tag{1.2}$$

with smooth functions $A_i(.)$, $z(x)$, $b(.)$, $A_i \in C^1(\mathrm{I\!R})$, $\mathrm{z} \in \mathrm{C}^1(\mathrm{I\!R}^{\mathrm{N}})$, $b \in C^1(\mathrm{I\!R})$, $\mathrm{b}(0) = 0$, $\|\mathrm{b}'\|_{\mathrm{L}^\infty} \leq \mathrm{K_b}$, $\mathrm{K_b} = \mathrm{cst}$, where the unknown function $u(t,x)$ belongs to $\mathrm{I\!R}$. Also, the equation (1.1) is endowed with the full family of entropy inequalities

$$\frac{\partial S(u)}{\partial t} + \sum_{i=1}^{N} \frac{\partial \eta_i(u)}{\partial x_i} + S'(u) b(u) \nabla z(x) \leq 0, \tag{1.3}$$

for all convex entropy functions $S(\cdot)$ and corresponding entropy fluxes $\eta_i(\cdot)$ defined in accordance with the relation

$$\eta_i\prime(u) = S'(u) a_i(u), \qquad a_i(u) = A_i'(u) \tag{1.4}$$

see Kruzkov [3], Lax [4] for more details.

The implicit schemes we develop here are linear at $(n+1)$-th time level and nonlinear at n-th time level with respect to grid function. The schemes possesse with a good stability property as purely implicit ones and are less expensive because of the linearity in the implicit part. Thus in some sense the new schemes have "averaged" properties of purely implicit and purely explicit schemes. The cost of these improved properties is that we can justify the convergence of the scheme for steady state calculations only.

The starting point in construction of the scheme is a kinetic formulation of scalar conservation laws by P.L.Lions,B.Perthame,E.Tadmor [5] that enables to rewrite (1.1)-(1.4) equivalently as kinetic equation with a kinetic "equilibrium" function $\chi(\xi; u)$

$$\frac{\partial \chi(\xi; u)}{\partial t} + \sum_{i=1}^{N} a_i(\xi) \cdot \frac{\partial \chi(\xi; u)}{\partial x_i} - b(\xi) \nabla z(x) \frac{\partial \chi(\xi; u)}{\partial \xi} = \frac{\partial m(t, x, \xi)}{\partial \xi} \tag{1.5}$$

for some nonnegative bounded measure $m(t, x, \xi)$ which satisfies

$$m(t, x, \xi) = 0 \quad \text{for} \quad |\xi| > \|u(t, \cdot)\|_{L^\infty}, \tag{1.6}$$

and

$$\chi(\xi; u) = \begin{cases} +1, & 0 < \xi \leq u, \\ -1, & u \leq \xi < 0, \\ \ \ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

The rest of the paper is organized as follows. We start construction of the scheme with homogenous equation, i.e. $z(x) = 0$, in single space dimension, $N = 1$. Then we develop the scheme on the basis of usual methodology of kinetic schemes + fixing of suitably selected kinetic velocities at implicit $(n+1)$-th time level in the usual purely implicit kinetic scheme. In section 2 we obtain desired apriori estimates (L^1, L^∞, BV bounds) on the family of approximate solutions and prove the convergence of time marching procedure for homogenous equation, i.e. the convergence towards steady state solutions as $t \longrightarrow \infty$. To our knowledge this is the first proof of convergence of the time marching procedure for scalar conservation laws. Notice that the extension of the developed implicit method for conservation laws with source terms is straightforward when space discretization is specified, see remarks 2.1, 2.3 and 2.5. In section 3 we consider scalar conservation laws with stiff source term. For numerical solution of this problem we modify artificial viscosity coefficient in the standard Engquist-Osher scheme and select the approximation for the source term and space derivatives in such a way that the resulting scheme is exact on the equilibriums, i.e. steady state solutions. We emphasize that after this modification artificial viscosity coefficients become discontinuous and discretization of the spatial derivatives is reduced to the central finite differences close by equilibriums. Evidently, resulting space discretization is stiff and this is confirmed by numerical tests, see section 4. Since implicit kinetic method has sufficiently large reserve of stability, introducing of it in modified scheme results in efficient algorithm. Numerical tests in section 4 show advantages of this implicit scheme over the standard ones. For the simplicity of exposition we restrict ourselves by one dimensional in space case making at the same time the remarks regarding extensions in multi dimension.

2 Implicit Kinetic Schemes for homogenous equation

We start construction of the schemes for homogenous equation ($z = 0$) in one space dimension ($N = 1$). Therefore we drop the subscripts and the source term where appropriate below.

2.1 Purely Implicit Schemes

A certain class of purely implicit schemes can be constructed by means of discretization of (1.5), e.g:

$$\begin{aligned}\chi_j^{n+1}(\xi) - \chi_j^n(\xi) + \frac{\Delta t}{\Delta x}\Big(a_-(\xi)\chi_{j+1}^{n+1}(\xi) + a_+(\xi)\chi_j^{n+1}(\xi) \\ -a_-(\xi)\chi_j^{n+1}(\xi) - a_+(\xi)\chi_{j-1}^{n+1}(\xi)\Big) = \frac{\partial m_j^{n+1}}{\partial \xi,}\end{aligned} \tag{2.1}$$

where

$$\chi_j^{n+1}(\xi) = \chi_{u_j^{n+1}}(\xi) \tag{2.2}$$

$$u_j^{n+1} = \int \chi_j^{n+1}(\xi) d\xi \tag{2.3}$$

$$a(\xi) = a_-(\xi) + a_+(\xi), \quad , a_+(\xi) \geq 0, \quad a_-(\xi) \leq 0, \tag{2.4}$$

$$m_j^{n+1}(\xi) \textit{ is some bounded nonnegative compactly supported measure.} \tag{2.5}$$

Notice that for construction of the scheme at macroscopic level we do not need the exact expression for $m_j^{n+1}(\xi)$ and knowledge of (2.5) is sufficient. Indeed, integrating of (2.1) in ξ yields

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}\Big(A(u_{j+1}^{n+1}, u_j^{n+1}) - A(u_j^{n+1}, u_{j-1}^{n+1})\Big) = 0. \tag{2.6}$$

where

$$A(u,v) = \int_0^u a_-(\xi)\, d\xi + \int_0^v a_+(\xi)\, d\xi \tag{2.7}$$

is usual Engquist-Osher numerical flux function [6]. Numerical schemes in the form (2.6) with monotone numerical flux functions are studied by several authors, see e.g. Sanders [8]. As is well known the numerical solutions of these schemes satisfy the estimates

$$\min_j(u_j^n) \leq u_j^{n+1} \leq \max_j(u_j^n), \tag{2.8}$$

$$\sum_j |u_j^{n+1} - u_{j-1}^{n+1}| \leq \sum_j |u_j^n - u_{j-1}^n|, \tag{2.9}$$

$$\sum_j |u_j^{n+1}|\Delta x \leq \sum_j |u_j^n|\Delta x, \tag{2.10}$$

$$S(u_j^{n+1}) - S(u_j^n) + \frac{\Delta t}{\Delta x}\Big(\eta(u_{j+1}^{n+1}, u_j^{n+1}) - \eta(u_j^{n+1}, u_{j-1}^{n+1})\Big) \leq 0. \tag{2.11}$$

where S is entropy function and corresponding numerical entropy flux is defined as

$$\eta(u, v) = \int_0^u S'(\xi)a_-(\xi)\, d\xi + \int_0^v S'(\xi)a_+(\xi)\, d\xi. \tag{2.12}$$

.

2.2 Construction of Implicit Kinetic Schemes

The main drawback of implicit scheme (2.6) is that it is nonlinear at $(n+1)$-th time level and thus very expensive from the standpoint of computational costs.To smooth this restriction we use equivalent kinetic reformulation (2.1) for implicit scheme (2.6) and suppose to select in a suitable way kinetic velocities and to fix corresponding propagation speeds in the implicit scheme (2.1) written at kinetic level. Thus setting

$$\xi_{+,j-1/2}^n = const, \quad \xi_{-,j+1/2}^n = const, \tag{2.13}$$

$$a_+(\xi_{+,j-1/2}^n) = a_{+,j-1/2}^n, \quad a_-(\xi_{-,j+1/2}^n) = a_{-,j+1/2}^n, \tag{2.14}$$

and fixing kinetic velocities in (2.1) we arrive at the following implicit scheme at kinetic level:

$$\begin{aligned} \chi_j^{n+1}(\xi) - \chi_j^n(\xi) + \frac{\Delta t}{\Delta x}\Big(a_{-,j+1/2}^n\chi_{j+1}^{n+1}(\xi) + a_{+,j-1/2}^n\chi_j^{n+1}(\xi) \\ -a_{-,j+1/2}^n\chi_j^{n+1}(\xi) - a_{+,j-1/2}^n\chi_{j-1}^{n+1}(\xi)\Big) = \frac{\partial\tilde{m}_j^{n+1}}{\partial\xi.} \end{aligned} \tag{2.15}$$

Integration of (2.15) in ξ results in the following implicit kinetic scheme at macroscopic level:

$$\begin{aligned}\tilde{u}_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}\Big(& a^n_{-,j+1/2}\tilde{u}^{n+1}_{j+1} + a^n_{+,j-1/2}\tilde{u}^{n+1}_j \\ & -a^n_{-,j+1/2}\tilde{u}^{n+1}_j - a^n_{+,j-1/2}\tilde{u}^{n+1}_j\Big) = 0.\end{aligned} \tag{2.16}$$

Lemma 2.1 Solutions of (2.16) satisfy the estimates (2.8)-(2.10).

Proof of the *lemma 2.1* can be performed on the basis of the standard technique used for purely implicit schemes, see e.g. [8], and therefore we omit it.

For the convenience of further exposition let denote via $L(\xi)$ and $L(\xi^*)$ linear operators corresponding to space discretizations in (2.1) and (2.15) respectively. Then (2.1) and (2.15) are written equivalently as

$$(I + \lambda L(\xi))f^{n+1} = f^n + \frac{\partial m^{n+1}}{\partial \xi,} \tag{2.17}$$

$$(I + \lambda L(\xi_*^{n+1}))\tilde{f}^{n+1} = f^n + \frac{\partial \tilde{m}^{n+1}}{\partial \xi,} \tag{2.18}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, and

$$f^{n+1} = \{f_j^{n+1}(\xi)\}_j, \quad \tilde{f}^{n+1} = \{f_j^{n+1}(\xi_*^{n+1})\}_j$$

denote the vector-functions corresponding to solution of (2.1),(2.15) respectively, $m^{n+1}, \tilde{m}^{n+1}$ are corresponding measure-valued vectors, ξ is usual kinetic velocity, ξ_*^{n+1} is a set of the fixed kinetic velocities, $\xi^* = (\xi_+^*, \xi_-^*)$. $\xi_+^*, \xi_-^* \in l^1$, see remark 2.1 for certain choice of ξ_*^{n+1}. Notice that we can consider operators $L(\xi), L(\xi^*)$ as a limit of $(2k+1) \times (2k+1)$ matrixes as $k \longrightarrow \infty$,

$$L_k(\xi) = tridiag\{-a^+(\xi), a^+(\xi) - a^-(\xi), a^-(\xi)\}_{j=-k}^k$$

$$L_k(\xi_*^{n+1}) = tridiag\{-a^+(\xi^{n+1}_{+,j-1/2}), a^+(\xi^{n+1}_{+,j-1/2}) - a^-(\xi^{n+1}_{-,j+1/2}), a^-(\xi^{n+1}_{-,j+1/2})\}_{j=-k}^k.$$

Lemma 2.2 If $(I + \lambda L(\xi_*^{n+1}))^{-1}$ exists and

$$\int L(\xi_*^{n+1}) f^n(\xi) d\xi = \int L(\xi) f^n(\xi) d\xi \tag{2.19}$$

then implicit kinetic scheme (2.16) is consistent with (1.1) in the sense of local truncation error.

Proof of Lemma 2.2 Equations (2.17),(2.18) can be written in equivalent form as

$$(I+\lambda L(\xi))\Delta f^{n+1}(\xi) = -\lambda L(\xi) f^n(\xi) + \frac{\partial m^{n+1}}{\partial \xi}, \tag{2.20}$$

$$(I+\lambda L(\xi_*^{n+1}))\Delta \tilde{f}^{n+1}(\xi) = -\lambda L(\xi_*^{n+1}) f^n(\xi) + \frac{\partial \tilde{m}^{n+1}}{\partial \xi}, \tag{2.21}$$

$$\Delta f^{n+1}(\xi) = f^{n+1}(\xi) - f^n(\xi), \qquad \Delta \tilde{f}^{n+1}(\xi) = \tilde{f}^{n+1}(\xi) - f^n(\xi).$$

With account of the existence of $(I+\lambda L(\xi_*^{n+1}))^{-1}$ (2.21) can be rewritten in the following equivalent form

$$\Delta \tilde{f}^{n+1}(\xi) = (I+\lambda L(\xi_*^{n+1}))^{-1}\lambda L(\xi_*^{n+1}) f^n(\xi) + (I+\lambda L(\xi_*^{n+1}))^{-1}\frac{\partial \tilde{m}^{n+1}}{\partial \xi}. \tag{2.22}$$

We integrate (2.22) in ξ. Then consecutive substitutions by using of (2.19), (2.20) in the right hand side of it yield:

$$\begin{aligned} \Delta \tilde{u}^{n+1} &= -(I+\lambda L(\xi_*^{n+1}))^{-1}\int \lambda L(\xi_*^{n+1}) f^n(\xi) d\xi = \\ &-(I+\lambda L(\xi_*^{n+1}))^{-1}\int (I+\lambda L(\xi))\Delta f^{n+1}(\xi) d\xi. \end{aligned} \tag{2.23}$$

Using classical representation of functions from matrixes in the form of series one can write:

$$(I+\lambda L(\xi_*^{n+1}))^{-1} = \sum_{i=0}^{\infty}(-1)^i \lambda^i L^i(\xi_*^{n+1}). \tag{2.24}$$

Substitution of (2.24) in right hand side of (2.23) results in the relation between solutions of equations (2.20) and (2.21):

$$\Delta \tilde{u}^{n+1} = \Delta u^{n+1} + \lambda \int (I+\lambda L(\xi_*^{n+1}))^{-1}(L(\xi) - L(\xi_*^{n+1}))\Delta f^{n+1}(\xi) d\xi. \tag{2.25}$$

Following standard techniques for calculation of local truncation errors, i.e. assuming sufficient smoothness of f^{n+1} and substituting it's Taylor's expansion in the expression under consideration results with account of the structures of $L(\xi)$ and $L(\xi_*)$ in the estimate

$$\lambda \int (I + \lambda L(\xi_*^{n+1}))^{-1}(L(\xi) - L(\xi_*^{n+1})\Delta f^{n+1}(\xi) d\xi = O(\Delta t \Delta x). \tag{2.26}$$

Since purely implicit scheme (2.1) is first order accurate in the sense of local truncation error, clearly, with account of (2.25) and (2.26) one has the same first order truncation error for implicit kinetic scheme (2.16) and this concludes the proof.

Remark 2.1 It is easy to see the existence of values $\xi_{+,j-1/2}^{n+1}$, $\xi_{-,j+1/2}^{n+1}$, e.g. by the following selections:

$$a_+(\xi_{+,j-1/2}^n) = \begin{cases} \dfrac{A^+(u_j^n) - A^+(u_{j-1}^n)}{u_j^n - u_{j-1}^n}, & \text{if } u_j^n \neq u_{j-1}^n \\ a_+(\theta u_j^n + (1-\theta)u_{j-1}^n), & \text{otherwise.} \end{cases}$$

$$a_-(\xi_{-,j+1/2}^n) = \begin{cases} \dfrac{A^-(u_{j+1}^n) - A^-(u_j^n)}{u_{j+1}^n - u_j^n}, & \text{if } u_{j+1}^n \neq u_j^n \\ a_-(\tilde{\theta} u_{j+1}^n + (1-\tilde{\theta})u_j^n), & \text{otherwise.} \end{cases}$$

for some $\theta, \tilde{\theta}$, $0 \leq \theta, \tilde{\theta} \leq 1$. In case of several independent space variables, i.e. $N \geq 2$, similar formulaes can be used for selection of ξ^* and thus the extension of implicit kinetic schemes for the equation in multi dimension is straightforward from this viewpoint as well. Namely, following the approach stated above in case of several independent space variables implicit kinetic scheme writes:

$$\tilde{u}_j^{n+1} - u_j^n + \frac{\Delta t}{\text{area}(C_j)} \sum_\alpha \text{area}(\Gamma_{j\alpha}) a_{j\alpha}^{\tilde{n}} (u_{j\alpha}^n - u_j^n) = 0,$$

where C_j is a cell with center $x_j \in \mathbb{R}^{\mathrm{N}}$, α runs a set of indices corresponding to surrounding x_j nodes, $\Gamma_{j\alpha}$ is cell interface between cells C_j and C_α,

$$\tilde{a}_{j\alpha}^n = \frac{\int_{\mathbb{R}_\xi^{\mathrm{n}}} < a(\xi), \overrightarrow{n}_{j\alpha} >_- (\chi_{j\alpha}^n(\xi) - \chi_j^n(\xi)) d\xi}{u_{j\alpha}^n - u_j^n},$$

$$< a(\xi), \overrightarrow{n} >= \min\,(\sum_{i=1}^N a_i(\xi) n_i,\ 0),$$

$\vec{n}_{j\alpha}$ is outward normal vector to cell interface between C_j and C_α.

Remark 2.2 Implicit kinetic scheme (2.16) is not conservative though in case of sufficient smoothness of the solution corresponding approximate solution by implicit kinetic scheme is close by to the one by conservative purely implicit scheme (2.1), see (2.25),(2.26).

Remark 2.3 One of the important properties of matrices $L_k(\xi), L_k(\xi_*^{n+1})$ used in the next section for further analysis of implicit kinetic schemes is that the sum of the elements in the rows is zero except of the first and last ones that e.g. for $L_k(\xi_*^{n+1})$, are equal to $a_+(\xi_{+,1/2-k}^{n+1})$ and $a_-(\xi_{-,k-1/2}^{n+1})$ respectively. Clearly, in multidimensional case the matrixes $L_k(\xi), L_k(\xi_*^{n+1})$ will not be tridiagonal as above and they can have more complicated block structure on unstructured grids. But even in that case the property mentioned above remains valid: the sum of the elements in the rows is nonnegative and it is zero except of those ones corresponding to the "boundaries".

Remark 2.4 Derivation of formulae (2.26) as it is presented in the proof of lemma 2.1 is formal since the equilibrium function at kinetic level is discontinuous in ξ. But it is easy to see that on smooth solutions of scalar conservation laws under consideration the result remains true, e.g. perform Taylor's expansions in (2.26) after integration in ξ.

2.3 Convergence towards steady state solutions

In case of $\Delta\tilde{u}^{n+1} = 0$ implicit kinetic scheme (2.16) results in conservative finite difference approximation for the steady state equation

$$\sum_{i=1}^{N} \frac{\partial A_i(u)}{\partial x_i} = 0, \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{2.27}$$

Clearly, corresponding to (2.27) entropy inequality is the following

$$\sum_{i=1}^{N} \frac{\partial \eta_i(u)}{\partial x_i} \leq 0. \tag{2.28}$$

We will prove the convergence of implicit kinetic schemes towards steady state when $t \longrightarrow \infty$. First of all we prove the convergence of the time marching procedure for (2.16), i.e.

$$\lim_{n\to\infty} \Delta\tilde{u}^{n+1} = 0 \tag{2.29}$$

for Δt, Δx fixed. To do so we need to recall logarithmic norm of a matrix and some properties of it, see e.g [2] and references therein. Namely

$$\mu_1[B] = max_j(b_{jj} + \sum_{i\neq j} |b_{ij}|) \tag{2.30}$$

$$\|By\| \geq \max(-\mu_1[-B], -\mu_1[B])\|y\|, \tag{2.31}$$

where via $\mu_1[B]$ the logarithmic norm of a matrix B induced by the usual vector norm

$$\|y\|_1 = \sum_j |y_j|$$

is denoted. Application of (2.30)-(2.31) with respect to finite dimensional linear operator that is of similar structure with the one presented in implicit kinetic scheme (2.18) yields:

$$\begin{array}{l} -\mu_1[-(I+\lambda L_k(\xi_*^{n+1}))] = 1 + \lambda max(a^{n+1}_{+,-k+1/2}, -a^{n+1}_{-,k-1/2}), \\ \mu_1[-(I+\lambda L_k(\xi_*^{n+1}))] < 0, \\ \mu_1[(I+\lambda L_k(\xi_*^{n+1}))y_k] \geq (1 + \lambda max(a^{n+1}_{+,-k+1/2}, -a^{n+1}_{-,k-1/2}))\|y_k\|_1, \end{array} \tag{2.32}$$

for any vector $y_k \in E^{2k+1}$. Pointing out from (2.32) we arrive at the following desired estimate

$$\|y_k\|_1 \leq \frac{\|(I+\lambda L_k(\xi_*^{n+1}))y_k\|}{1+\lambda \max(a^{n+1}_{+,-k+1/2}, -a^{n+1}_{-,k-1/2})}$$

and thus

$$\|(I+\lambda L_k(\xi_*^{n+1}))^{-1}\| \leq (1+\lambda \max(a^{n+1}_{+,-k+1/2}, -a^{n+1}_{-,k-1/2}))^{-1}.$$

Clearly, since $y \in l^1$ passing to the limit in (2.32) will result in similar estimates for $k=\infty$, i.e. for the linear operator of implicit kinetic scheme (2.18). These estimates write:

$$\|y\|_1 \leq \frac{\|(I+\lambda L(\xi_*^{n+1}))y\|}{1+\lambda \max(a_+(0), -a_-(0))}. \tag{2.33}$$

$$\|(I+\lambda L(\xi_*^{n+1}))^{-1}\|_1 \leq (1+\lambda \max(a_+(0), -a_-(0)))^{-1}. \tag{2.34}$$

Notice that in general the denominator in the right hand side of (2.33) can be reduced to 1 thus making the estimate useless in the proof of (2.29). One can avoid this difficulty simply by means of selection of splitting (2.4) in such a way that

$$a_+(0) \neq 0, \quad a_-(0) \neq 0. \tag{2.35}$$

Multiplying of (2.18) on $(I + \lambda L(\xi_*^n))$ yields:

$$\begin{aligned} &(I + \lambda L(\xi_*^n))(I + \lambda L(\xi_*^{n+1}))\tilde{f}^{n+1} = \\ &\tilde{f}^{n-1} + \frac{\partial m_j^n}{\partial \xi} + (I + \lambda L(\xi_*^n))\frac{\partial m^{n+1}}{\partial \xi} \end{aligned} \tag{2.36}$$

Repeating the same procedure with respect to (2.36), i.e. multiplying on $(I + \lambda L(\xi_*^{n-1}))$ etc., one arrives at the following equivalent form of implicit kinetic scheme (2.18):

$$\prod_{i=1}^{n+1}(I + \lambda L(\xi_*^i))\tilde{f}^{n+1} = \tilde{f}^0 + \sum_{k=0}^{n}\prod_{i=1}^{k}(I + \lambda L(\xi_*^i))\frac{\partial m^{k+1}}{\partial \xi},$$

$$\begin{aligned} &\prod_{i=1}^{n+1}(I + \lambda L(\xi_*^i))(\tilde{f}^{n+1} - \tilde{f}^n) = \\ &\tilde{f}^0 - \prod_{i=1}^{n+1}(I + \lambda L(\xi_*^i))\tilde{f}^n + \sum_{k=0}^{n}\prod_{i=1}^{k}(I + \lambda L(\xi_*^i))\frac{\partial m^{k+1}}{\partial \xi} = \\ &\tilde{f}^0 - (I + \lambda \tilde{L}(\xi_*^{n+1}))\prod_{i=1}^{n}(I + \lambda L(\xi_*^i))\tilde{f}^n + \sum_{k=0}^{n}\prod_{i=1}^{k}(I + \lambda L(\xi_*^i))\frac{\partial m^{k+1}}{\partial \xi} = \\ &\tilde{f}^0 - (I + \lambda \tilde{L}(\xi_*^{n+1}))(\tilde{f}^0 + \sum_{k=0}^{n-1}\prod_{i=1}^{k}(I + \lambda L(\xi_*^i))\frac{\partial m^{k+1}}{\partial \xi}) + \\ &\sum_{k=0}^{n}\prod_{i=1}^{k}(I + \lambda L(\xi_*^i))\frac{\partial m^{k+1}}{\partial \xi}, \end{aligned} \tag{2.37}$$

where

$$\tilde{L}(\xi_*^{n+1}) = \prod_{i=1}^{n}(I + \lambda L(\xi_*^i)) \cdot L(\xi_*^{n+1}) \cdot \prod_{i=1}^{n}(I + \lambda L(\xi_*^i))^{-1},$$

$$\prod_{i=1}^{k}(I + \lambda L(\xi_*^i)) = (I + \lambda L(\xi_*^1)) \cdot (I + \lambda L(\xi_*^2)) \cdots (I + \lambda L(\xi_*^k)).$$

Integration of (2.37) in ξ results in implicit kinetic scheme in desired Δ-form:

$$\begin{aligned} &\prod_{i=1}^{n+1}(I + \lambda L(\xi_*^i))\Delta\tilde{u}^{n+1} = -\lambda\tilde{L}(\xi_*^{n+1})\tilde{u}^0, \\ &(I + \lambda L(\xi_*^i))\Delta\tilde{u}^{n+1} = -\lambda L(\xi_*^{n+1})\prod_{i=1}^{n+1}(I + \lambda L(\xi_*^i))^{-1}\tilde{u}^0. \end{aligned} \tag{2.38}$$

Notice that due to *lemma 2.1* $\tilde{u}^n \in L^1 \cap L^\infty$ for any $n \in \mathbb{N}$ and one can estimate

$$\|\lambda L(\xi_*^{n+1})\tilde{u}^k\| \leq 2\lambda \max_{|\xi|\leq\|u_0\|_\infty} (|a(\xi)|) \cdot \|\tilde{u}^k\|_1. \tag{2.39}$$

The L^1-norm of the left hand side of (2.38) can be estimated by means of using (2.34) and finally we arrive at the following estimate:

$$\|\Delta\tilde{u}^{n+1}\| \leq \frac{2\lambda \max_{|\xi|\leq\|u_0\|_\infty} (|a(\xi)|) \cdot \|\tilde{u}^0\|_1}{(1+\lambda max(a_+(0), -a_-(0)))^{n+1}}. \tag{2.40}$$

Clearly, with account of assumption (2.35) the estimate (2.40) results in (2.29).

As usual we define approximate solution $u_{\Delta x}$ by means of piecewise constant reconstruction on time$\times$space cells. Now we are ready to prove the following convergence theorem.

Theorem 2.3 If $(I+\lambda L(\xi_*^{n+1}))^{-1}$ exists for any $n \in \mathbb{N}$, (2.35) holds true and $z=0$ then there exists $u_{\Delta x}$, such a subsequence of approximate solutions constructed by implicit kinetic scheme (2.16) that in the limit $t \longrightarrow \infty$, $\Delta x \rightarrow 0$ converges towards some steady state entropy solution to (1.1),(1.2).

Proof of the Theorem 2.3. The proof consists of the following four steps
(i) Derivation of apriori estimates, see *lemma 2.1.*
(ii) Convergence of time marching procedure, i.e. passing to the limit $t \rightarrow \infty$, see (2.40).
(iii) Derivation of steady state entropy inequality.
Recall that purely implicit scheme (2.6) corresponding to initial data $\tilde{u}_j^n$ satisfies the following in-cell entropy inequality

$$S(u_j^{n+1}) - S(\tilde{u}_j^n) + \frac{\Delta t}{\Delta x}\Big(\eta(u_{j+1}^{n+1}, u_j^{n+1}) - \eta(u_j^{n+1}, u_{j-1}^{n+1})\Big) \leq 0. \tag{2.41}$$

One can obtain the analogue of (2.38) for $\tilde{u}^{n+1} - u^n$ by using the same technique as above; namely, one has

$$\prod_{i=1}^{n}(I+\lambda L(\xi_*^i)) \int (I+\lambda L(\xi))(\chi_{u_j^{n+1}}(\xi) - \chi_{\tilde{u}_j^n}(\xi))\, d\xi = -\int \lambda\tilde{L}(\xi)\chi_{u_0}(\xi)\, d\xi \tag{2.42}$$

where

$$\tilde{L}(\xi) = \prod_{i=1}^{n}(I+\lambda L(\xi_*^i)) \cdot L(\xi) \cdot \prod_{i=1}^{n}(I+\lambda L(\xi_*^i))^{-1},$$

and thus one can deduce from (2.42) that

$$\lim_{n\to\infty} \|u^{n+1} - \tilde{u}^n\|_{L^1} = 0. \tag{2.43}$$

Multiplying of (2.41) in $\Delta x \varphi_j$, $\varphi_j = \varphi(x_j)$, φ - nonnegative smooth test function, summing in j and passing to the limit $n \to \infty$ results thanks to (2.43) in desired steady state cell entropy inequality for the time marching procedure

$$\sum_j \varphi_j \Big(\eta(u^\infty_{j+1}, u^\infty_j) - \eta(u^\infty_j, u^\infty_{j-1}) \Big) \leq 0. \tag{2.44}$$

(iv) Extracting subsequence and passing to the limit.
The estimates (2.8)-(2.10) remain valid for the functions $u_{\Delta x}(x) = u^\infty_j, \quad x_{j-1/2} < x < x_{j+1/2}$, and thus one can extract the convergent a.e. subsequence. Performing the integration by parts in (2.44) and passing to the limit $\Delta x \to 0$, clearly, results in the validity of the steady state entropy inequality (2.28) in distributional sense for limiting function that concludes the proof.

Remark 2.5 Since $\lambda = \frac{\Delta t}{\Delta x}$ it is clear from (2.40) that large time steps can significantly accelerate the convergence towards steady state, e.g. compare the factors for different steps in time, see also Table 1. in section 4 and compare the errors for different CFL numbers. Note that for implicit kinetic schemes we do not need crucial CFL condition that is usual for explicit schemes. Instead we need less restrictive requirement - the existence of $(I + \lambda L(\xi^i_*))^{-1}$.
Remark2.6 In multidimensional case, since the matrixes corresponding to the implicit kinetic schemes have the suitable properties, see remarks 2.1,2.3, calculation of corresponding logarithmic norm for $(I + \lambda L(\xi^{n+1}_*))$ results in similar with (2.32) formulaes; thus one can arrive at similar with (2.40) estimate and as a result the convergence of the time marching procedure is ensured for implicit kinetic schemes in multi dimension as well.
Remark 2.7 If the uniqueness theorem for entropy steady state solutions would be available than the convergence of the implicit scheme under consideration follows by standard uniqueness arguments.

3 Equilibrium conserving implicit kinetic schemes for equation with stiff source term

3.1 Selection of equilibrium conserving discretization for space derivative and source term

Equation (1.1) admits steady state solutions defined by

$$D(u) + z(x) = \text{const}, \tag{3.1}$$

where

$$D(u) = \int_0^u \frac{a(s)}{b(s)} ds$$

and we assume that

$$0 < \frac{a(u)}{b(u)} < \infty, \quad D(\pm\infty) = \pm\infty. \tag{3.2}$$

The difficulty associated with numerical resolution of (1.1) is to preserve at a discrete level the equilibriums, i.e. steady state solutions given by (3.1), e.g. see [1] for the details and existing approaches. Below for (1.1) we introduce one more numerical scheme that is exact on the equilibriums. For the convenience of further exposition we consider again the Engquist-Osher scheme written in the viscosity form, see e.g. [7]. Notice that for the source term we use non standard approximation. The scheme writes:

$$\begin{aligned} &\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{A_{j+1}^n - A_j^n}{\Delta x} + \frac{A_j^n - A_{j-1}^n}{\Delta x} + \\ 0.5b_{j-1/2}^n &\frac{z_j - z_{j-1}}{\Delta x} + 0.5b_{j+1/2}^n \frac{z_{j+1} - z_j}{\Delta x} = \\ &0.5Q_{j+1/2}^n \frac{u_{j+1}^n - u_j^n}{\Delta x} - 0.5Q_{j-1/2}^n \frac{u_j^n - u_{j-1}^n}{\Delta x}, \end{aligned} \tag{3.3}$$

where $Q_{j\pm1/2}^n$ are artificial viscosity coefficients,

$$Q_{j+1/2}^n = \frac{|A_{j+1}^n| - |A_j^n|}{u_{j+1}^n - u_j^n}, \quad Q_{j-1/2}^n = \frac{|A_j^n| - |A_{j-1}^n|}{u_j^n - u_{j-1}^n}, \tag{3.4}$$

$$b_{j+1/2}^n = \frac{A_{j+1}^n - A_j^n}{D_{j+1}^n - D_j^n}, \quad b_{j-1/2}^n = \frac{A_j^n - A_{j-1}^n}{D_j^n - D_{j-1}^n}. \tag{3.5}$$

Lemma 3.1 Numerical scheme (3.3),(3.5) where artificial viscosity coefficient is set to zero is exact on the equilibriums.

Proof of Lemma 3.1 Rewriting of (3.1) for (x_{j-1}, x_j) and (x_j, x_{j+1}) yields

$$D(u_j) + z_j = D(u_{j-1}) + z_{j-1}, \quad D(u_j) + z_j = D(u_{j+1}) + z_{j+1} \tag{3.6}$$

Defining $z_{j+1} - z_j$ and $z_j - z_{j-1}$ from (3.6) and substituting them in (3.3) results in $u_j^{n+1} = u_j^n$, i.e. equilibrium initial data are maintained and this concludes the proof.

Clearly, artificial viscosity coefficients equal to zero reduce discretization of space derivative to central finite difference approximation that is known to be unstable. In order to maintain the stability and equilibrium conservation property we should ensure artificial viscosity coefficient to be zero close by equilibriums only. Thus we define

$$Q_{j+1/2}^n = \begin{cases} \dfrac{|A_{j+1}^n| - |A_j^n|}{u_{j+1}^n - u_j^n}, & \text{if } E_{j+1/2} > \varepsilon, \\ 0, & \text{otherwise}, \end{cases} \tag{3.7}$$

where $\varepsilon > 0$ is a small parameter the value of which will be specified later on from the viewpoint to ensure the convergence of the scheme, $E_{j+1/2}$ should characterize the deviation from the equilibrium and we define it according to formulae

$$E_{j+1/2} = |D(u_j) + z_j - D(u_{j+1}) - z_{j+1}|. \tag{3.8}$$

Introducing

$$\Theta_{j+1/2}^n = \begin{cases} 0, & \text{if } E_{j+1/2} < \varepsilon, \\ 1, & \text{otherwise}, \end{cases} \tag{3.9}$$

the scheme (3.3),(3.7),(3.8) writes in the equivalent but suitable for investigation form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \Theta_{j+1/2}^n \frac{A^-(u_{j+1}^n) - A^-(u_j^n)}{\Delta x} + \Theta_{j-1/2}^n \frac{A^+(u_j^n) - A^+(u_{j-1}^n)}{\Delta x} \tag{3.10}$$

$$+\Theta_{j-1/2}^n b_{j-1/2}^n \frac{z_j - z_{j-1}}{2\Delta x} + \Theta_{j+1/2}^n b_{j+1/2}^n \frac{z_{j+1} - z_j}{2\Delta x} = \tag{3.11}$$

$$(\Theta_{j-1/2}^n - 1) b_{j-1/2}^n \frac{D_j^n + z_j - D_{j-1}^n - z_{j-1}}{2\Delta x} + \tag{3.12}$$

$$(\Theta_{j+1/2}^n - 1) b_{j+1/2}^n \frac{D_{j+1}^n + z_{j+1} - D_j^n - z_j}{2\Delta x}. \tag{3.13}$$

One can observe that contribution to L^∞-norm of approximate solutions is provided by maximum principle for (3.10), the terms like (3.11) usually are controlled by $exp(TK_b\|z'\|_\infty)\|u_0\|_\infty$ and analogically (3.12),(3.13) can be controlled by $exp(TK_b\|z'\|_\infty\frac{\varepsilon}{\Delta x})\|u_0\|_\infty$. Based on these arguments exact computations result in the validity of following

Lemma 3.2 If $\varepsilon = \Delta x \Delta t^\gamma$, $\gamma \geq 0$, and CFL condition

$$\frac{\Delta t}{\Delta x} max_{|u|\leq K_\infty}|a(u)| \leq 1, \tag{3.14}$$

$$K_\infty = exp(TK_b\|z'\|_\infty(1+\Delta t^\gamma))\|u_0\|_\infty,$$

holds true then for any $t \leq T$ approximate solutions $u_{\Delta x}(t,x) = u_j^n$, $(t,x) \in (t_n, t_{n+1}) \times (x_{j-1/2}, x_{j+1/2})$ satisfy

$$\|u_{\Delta x}(t,x)\|_{L^\infty} \leq K_\infty,$$

$$\|u_{\Delta x}(t,x)\|_{L^1} \leq K_1, \quad K_1 = exp(TK_b\|z'\|_\infty(1+\Delta t^\gamma))\|u_0\|_{L^1}.$$

Remark 3.1 One of the drawbacks of the presented space discretization is that it results in the discontinuous artificial viscosity coefficient for numerical scheme; Another one is that it depends on thresholding parameter ε which although can be estimated from stability requirements according to *lemma 3.2* but at numerical level, as is well known, this can result in different results for different choices of ε in case of different initial data. Thus it is evident that the developed equilibrium conserving space discretization is stiff.

3.2 Coupling of implicit approach with equilibrium conserving discretization in space

In order to have consistent implicit time discretization we need the requirements of *lemma 2.2* to be satisfied. Simple computations verify that the following implicit scheme

$$\begin{aligned}\tilde{u}_j^{n+1} + \lambda\Big(&\Theta_{j+1/2}^n a_{-,j+1/2}^n \tilde{u}_{j+1}^{n+1} + \Theta_{j-1/2}^n a_{+,j-1/2}^n \tilde{u}_j^{n+1} \\ &-\Theta_{j+1/2}^n a_{-,j+1/2}^n \tilde{u}_j^{n+1} - \Theta_{j-1/2}^n a_{+,j-1/2}^n \tilde{u}_j^{n+1}\Big) \\ +\lambda \tilde{b}_{j+1/2}^n&(\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1}) + \lambda \tilde{b}_{j-1/2}^n(\tilde{u}_j^{n+1} - \tilde{u}_{j-1}^{n+1}) = u_j^n,\end{aligned} \tag{3.15}$$

where

$$\tilde{b}_{j+1/2}^n = \Big(z_{j+1} - z_j + (1-\Theta_{j+1/2}^n)(D_{j+1}^n - D_j^n)\Big) b_{j+1/2}^n (u_{j+1}^n - u_j^n)^{-1},$$

$$\tilde{b}^n_{j-1/2} = \Big(z_j - z_{j-1} + (1-\Theta^n_{j-1/2})(D^n_j - D^n_{j-1})\Big) b^n_{j-1/2}(u^n_j - u^n_{j-1})^{-1},$$

with $\Theta^n_{j\pm1/2}$, $b^n_{j\pm1/2}$ defined according to (3.7),(3.5) respectively, satisfies the requirement of *lemma 2.2* and thus (3.15) is consistent with scalar conservation law (1.1).

Remark 3.2 Evidently for implicit scheme (3.15) on finite time interval one can derive uniform L^1 and L^∞ estimates similar to the ones given in *lemma 3.2*. Notice that for this purpose the alternative to the classical technique could be the estimation of $\|(I+\lambda L(\xi^{n+1}_*))^{-1}\|$ by means of using corresponding logarithmic norm for $(I+\lambda L(\xi^{n+1}_*))$, by analogy as it was used in section 2 above.

4 Numerical test

In order to study computational capability of the developed implicit approach the following test problem is considered:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} + z'(x)u = 0, \tag{4.1}$$

$$u(x, t=0) = 0 \ \text{ for } x > 0, \qquad u(x=0, t) = 2 \ \text{ for } t > 0, \tag{4.2}$$

where the function $z(x)$ is choosen as

$$z(x) = \begin{cases} \cos(\pi x), & 4.5 \le x \le 5.5, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

The steady state solution of this problem is given by the simple relation

$$u + z = 2. \tag{4.4}$$

All numerical computations of this test problem are performed with 101 nodal points in space. Below the results of computations are given for Engquist-Osher scheme, modified Engquist-Osher scheme with discontinuous artificial viscosity coefficients and by implicit kinetic scheme coupled with stiff space discretization that was considered in section 3. From the figures 1.-5. one sees that standard Engquist-Osher scheme gives high errors on large domain $x \ge 4.5$; the scheme with discontinuous artificial viscosity coefficient is oscillatory for CFL number 0.8; it gives better results then Engquist-Osher scheme for CFL number 0.2 but significant errors are still presented in the large domain; stabilizing effect and high accuracy of the developed implicit approach can be clearly observed by comparison of figures 1,2,3 and fig.4,5, see *table 1* as well.

Analysis of the formulae (2.40) gives two ways for improving the accuracy of computations by the implicit kinetic scheme, namely:
(i) to increase the number of time iterations;
(ii) to use large time steps, see remark 2.5 as well.

Table 1. Comparison of numerical schemes

$Method$	$CFL - number$	$Time$	$L^\infty - error$	$L^1 - error$
$Engquist - Osher$	0.2	20	0.1650527	0.4880051
$Modified$	0.2	20	0.1125576	0.2450109
$Implicit$	0.2	20	$6.78402 \cdot 10^{-2}$	0.1066506
$Engquist - Osher$	0.8	20	0.1650536	0.4880109
$Modified$	0.8	20	1.74094	1.78926
$Implicit$	0.8	20	$3.61729 \cdot 10^{-2}$	$8.00558 \cdot 10^{-2}$
$Engquist - Osher$	2	40	7.85635	1.93917
$Modified$	2	40	$overflow$	
$Implicit$	2	40	$1.66893 \cdot 10^{-6}$	$7.84397 \cdot 10^{-6}$

Table 2. Implicit kinetic scheme, influence of CFL-number and time iterations

N	$CFL - number$	$Time$	$L^\infty - error$	$L^1 - error$	$Iterations$
1	0.2	20	$6.78402 \cdot 10^{-2}$	0.1066506	3000
2	0.8	20	$3.61729 \cdot 10^{-2}$	$8.00558 \cdot 10^{-2}$	750
3	0.8	30	$2.69651 \cdot 10^{-4}$	$1.11885 \cdot 10^{-3}$	1125
4	0.8	40	$2.14577 \cdot 10^{-6}$	$8.73804 \cdot 10^{-6}$	1500
5	2	40	$1.66893 \cdot 10^{-6}$	$7.84397 \cdot 10^{-6}$	600
6	4	40	$2.57492 \cdot 10^{-5}$	$1.70803 \cdot 10^{-4}$	300
7	8	40	$5.99766 \cdot 10^{-3}$	$2.42001 \cdot 10^{-2}$	150
8	10	50	$6.22369 \cdot 10^{-3}$	$3.95213 \cdot 10^{-2}$	150
9	12	60	$2.80309 \cdot 10^{-3}$	$9.71185 \cdot 10^{-3}$	150
10	14	70	$1.44053 \cdot 10^{-3}$	$5.00364 \cdot 10^{-3}$	150
11	14	100	$7.15256 \cdot 10^{-7}$	$3.37362 \cdot 10^{-6}$	214

These conclusions are confirmed by calculations of the test problem for different CFL numbers and numbers of iterations in time, see *table 2*. In particular better results can be obtained by increasing number of iterations in time, e.g. compare the rows 5-7 in *table 2*, or by enlarging the time interval while keeping CFL number fixed, see rows 2-4 in *table 2*; For fixed number of iterations increasing of CFL number results in improvement of the accuracy as well , compare the rows 7-10 in *table 2*. Notice that suitable selection of the couple - CFL number, number of iteration in time - gives best numerical results, e.g. see the rows 1,2 and rows 4-6, *table 2*. Finally notice that though the computational cost per iteration for implicit scheme is higher then by the explicit schemes it is computationally less expensive and much more accurate then explicit schemes, e.g. compare the errors after 3000 time iterations by explicit Engquist-Osher scheme, see *table 1*, row 1, with 214 time iterations by implicit kinetic scheme, see *table 2*, last row.

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References

[1] Botchorishvili R., Perthame B., Vasseur A., Equilibrium schemes for scalar conservation laws with stiff sources, RR INRIA-8931 (2000), 1-55.

[2] Dekker K., Verwer J., Stability of Runge-Kutta methods for stiff nonlinear differential equations, North Holland, (1984), 1-307.

[3] Kruzkov S.N., Generalized solutions of the Cauchy problem in the large for nonlinear equations of first order, Dokl. Akad. Nauk. SSSR **187**(1) (1970) 29–32; English trans. in Soviet Math. Dokl. **10** (1969) .

[4] Lax P., Shock waves and entropy, in: "Contributions to Nonlinear Functional Analysis." E.H. Zarantonello, ed. New York: Academic Press (1971) 603–634.

[5] Lions P.L., Perthame B., Tadmor E., A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. **7** (1994) 169–191.

[6] Engquist B., Osher S., Stable and entropy satisfying approximations for transonic flow calculations, Math. Comp. **34** (1980) 45–75.

[7] Osher S., Tadmor E.,On the convergence of difference approximation to scalar conservation laws, Math.Comp., **50**, (1988) 19-51.

[8] Sanders R., On the convergence of monotone finite difference schemes with variable spatial differencing, Math.Comp., **40** (161), (1983) 91-106.

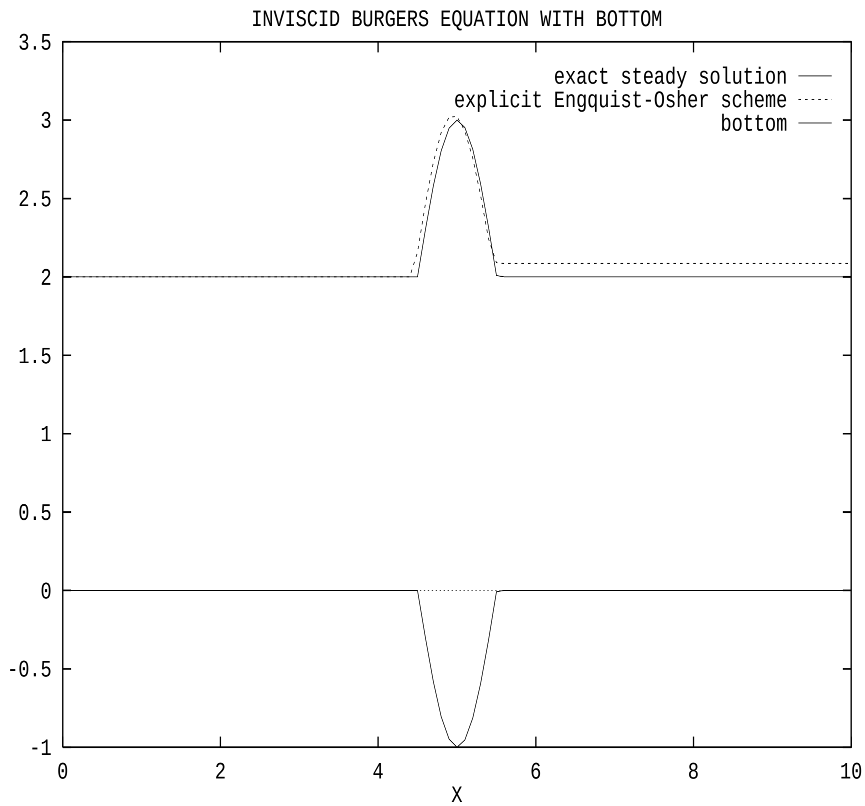

Figure 1: 101 nodes, CFL=0.75, time=20.

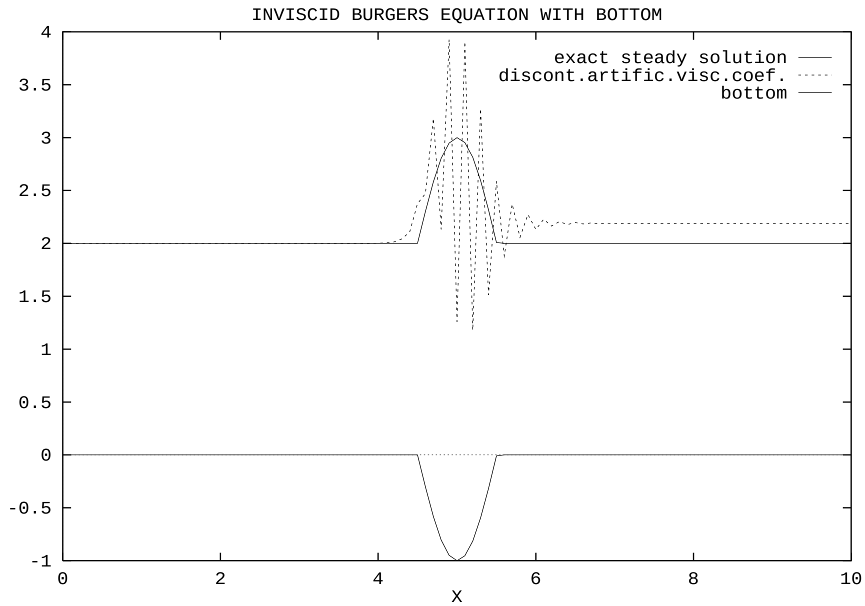

Figure 2: 101 nodes, CFL=0.8, time=20.

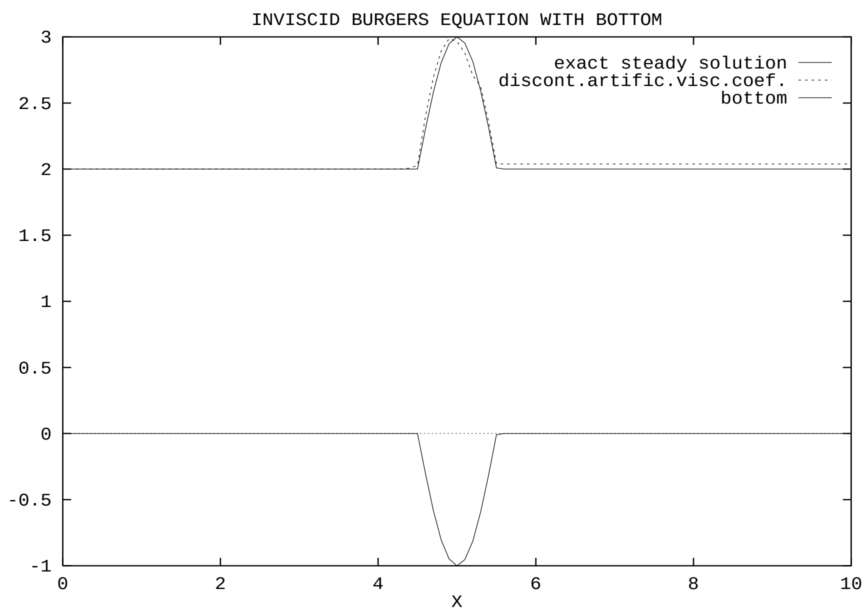

Figure 3: 101 nodes, CFL=0.2, time=20.

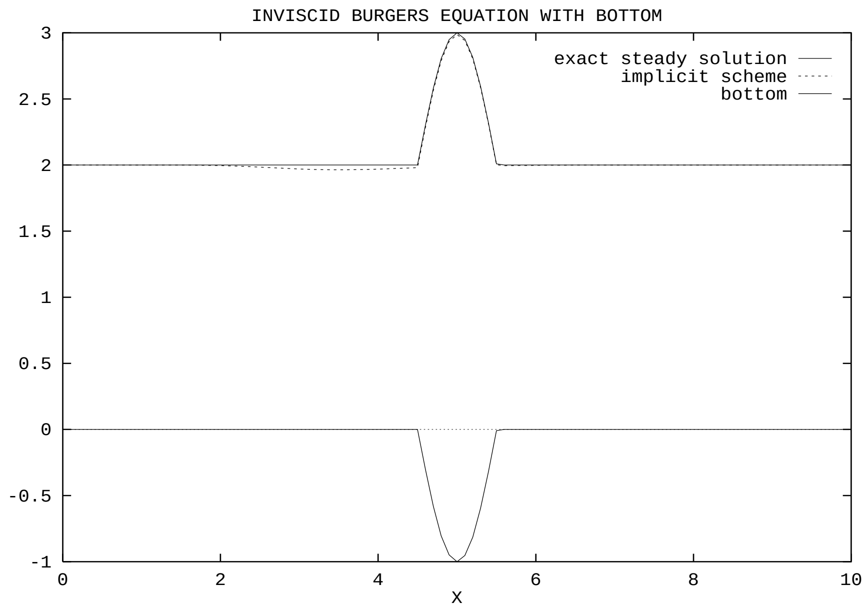

Figure 4: 101 nodes, CFL=0.8, time=20.

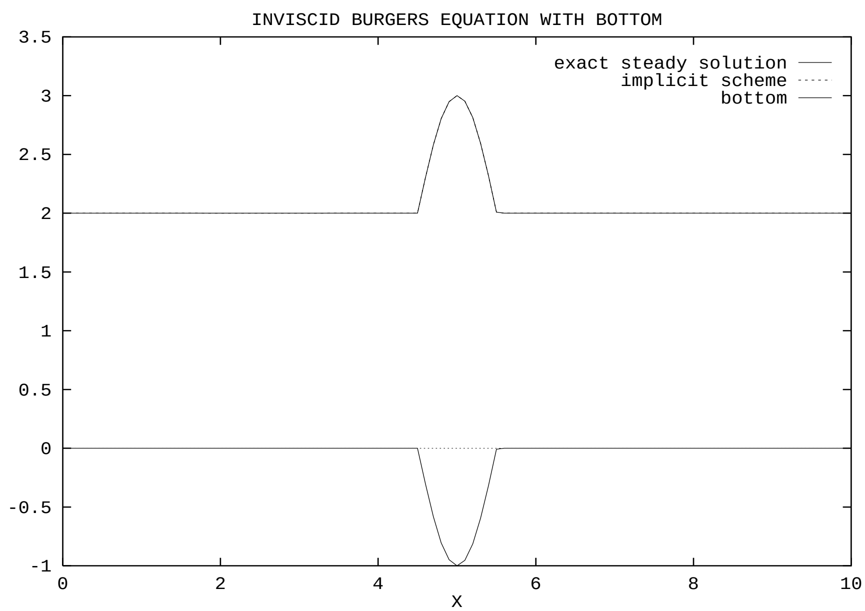

Figure 5: 101 nodes, CFL=2, time=40.

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